[Generic Competency - 5% of the final grade of the subject]

a) Expresseu el nombre 10.875 en base 2.

b) Es vol emmagatzemar el nombre \((1101.0110101)_2\) utilitzant un emmagatzematge en base 2, coma flotant, utilitzant una mantissa de 7 bits (un d’ells pel signe) i un exponent de 5 bits (un d’ells pel signe) i mitjançant arrodoniment per eliminació. Retorna la representació d’aquest nombre en aquest format i calcula quantes xifres significatives correctes (correct significant digits) té l’aproximació.

a) Donat el nombre 10.875, convertirem primer la part entera a binari i després la part decimal.

\[
\begin{align*}
10 & = 2 \times 5 + 0 \\
5 & = 2 \times 2 + 1 \\
2 & = 2 \times 1 + 0 \\
1 & = 2 \times 0 + 1 \\
\end{align*}
\]

Per tant \(10 = (1010)_2\). En relació a la part decimal

\[
\begin{align*}
0.875 \times 2 & = 0.75 + \frac{1}{2} \\
0.75 \times 2 & = 0.5 + \frac{1}{2} \\
0.5 \times 2 & = 0 + \frac{1}{2} \\
\end{align*}
\]

És a dir, \(0.875 = (0.111)_2\). Ajuntant els dos resultats anteriors, es té que \(10.875 = (1010.111)_2\).

b) Com que \((1101.0110101)_2 = (0.11010110101)_2 \cdot 2^4 = (0.11010110101)_2 \cdot 2^{(100)_2}\), si utilitzem 7 bits per a la mantissa i 5 per l’exponent i arrodonim mitjançant eliminació, guardaríem el següent nombre

\[
\begin{array}{c c c c c c c c}
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
\text{mantissa} & & & & & & & \\
\end{array}
\]

\[
\begin{array}{c c c c c}
0 & 1 & 0 & 1 & 0 \\
\text{exponent} & & & & \\
\end{array}
\]

que representa \((0.110101)_2 \cdot 2^{(100)_2} = (0.110101)_2 \cdot 2^4 = (1101.01)_2\).

Per saber el nombre de xifres significatives correctes, calculem primer

\[
\begin{align*}
(1101.0110101)_2 & = 2^3 + 2^2 + 2^0 + 2^{-2} + 2^{-3} + 2^{-5} + 2^{-7} + 2^{-9} = 13.416015625 \\
(1101.01)_2 & = 2^3 + 2^2 + 2^0 + 2^{-2} = 13.25
\end{align*}
\]

per tant, l’error relatiu comès és

\[
\frac{|13.416015625 - 13.25|}{13.416015625} = 0.012 = 0.12 \cdot 10^{-1} < 0.5 \cdot 10^{-1}.
\]

Per tant, l’aproximació té 1 xifra significativa correcta.

1. [3.5 points] We want to determine the intersection of the curves \(f(x) = 10 \sin(x)\) and \(g(x) = 10 - x^2\) in the interval \([0, 6]\).

a) [1 point] Do three iterations of the Newton method taking as initial guess \(x_0 = 0\).

b) [1 point] Compute the approximated relative error associated to \(x_2\) in absolute value.

c) [0.5 points] Using the previous results, and assuming that Newton method has the standard convergence rate, predict the approximated relative error associated to \(x_3\).

d) [1 point] Determine the intersection of the curves in \([0, 6]\). Compute this approximation and stop when in two consecutive iterations the first four decimals are not modified.
a) In order to find the intersection of the curves we must find the roots of

\[ h(x) = f(x) - g(x) = 10 \sin(x) - 10 + x^2 \]

The derivative of this function is:

\[ h'(x) = 10 \cos(x) + 2x \]

Let’s do three iterations of Newton method taking as initial guess \( x_0 = 0 \):

\[ x_1 = x_0 - \frac{h(x_0)}{h'(x_0)} = 0 - \frac{-10}{10} = 1 \]

\[ x_2 = x_1 - \frac{h(x_1)}{h'(x_1)} = 1.079060965673297 \]

\[ x_3 = x_2 - \frac{h(x_2)}{h'(x_2)} = 1.082037727741138 \]

b) The approximate relative error associated to \( x_2 \) in absolute value is:

\[ |\tilde{r}_2| = \frac{|x_3 - x_2|}{|x_3|} = 0.002751070495532 \]

c) Let’s assume that Newton has the standard convergence rate, then:

\[ |\tilde{r}_3| \approx \lambda |\tilde{r}_2|^2 \approx |0.002751070495532|^2 = 7.5684 \cdot 10^{-6} \]

d) Let’s continue the iterations with Newton method:

\[ x_4 = x_3 - \frac{h(x_3)}{h'(x_3)} = 1.082042132751058 \]

The first three decimals have not been modified with respect to \( x_3 \).

2. [3 points] The data below represents the bacterial growth in a liquid culture over a number of days.

<table>
<thead>
<tr>
<th>Day</th>
<th>Amount \times 10^6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>67</td>
</tr>
<tr>
<td>8</td>
<td>98</td>
</tr>
<tr>
<td>16</td>
<td>149</td>
</tr>
</tbody>
</table>

a) [1 point] Find the pure interpolating polynomial using all the data points. Use Lagrange polynomials.

b) [1 point] Find the linear Spline that interpolates the given data. Use the shape functions \( \Phi_i \).

c) [1 point] We want to approximate with the least squares criterion the given data using an interpolant of the form

\[ p(x) = e^{\beta x}. \]

Write down the equation to solve in order to find the value of \( \beta \). Substitute the values of the given data and simplify the most you can the equation. Is this equation analytically solvable?

Solution

a) Let’s define:

\[ x_0 = 0, x_1 = 8, x_2 = 16 \]

then the pure interpolating polynomial using Lagrange polynomials is

\[ p_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x) = 67L_0(x) + 98L_1(x) + 149L_2(x) \]

where

\[ L_0(x) = \frac{(x-8)(x-16)}{(-8)(-16)} = \frac{(x-8)(x-16)}{128} \]

\[ L_1(x) = \frac{(x-0)(x-16)}{(8-0)(8-16)} = \frac{-x(x-16)}{64} \]

\[ L_2(x) = \frac{(x-0)(x-8)}{(16-0)(16-8)} = \frac{x(x-8)}{128} \]
b) Using the Lagrange polynomial
\[ L_0^0 = \frac{x - 8}{0 - 8} = \frac{8 - x}{8} \]
the \( \Phi_0(x) \) basis function is defined as
\[ \Phi_0(x) = \begin{cases} \frac{8 - x}{8} & x \in [0, 8] \\ 0 & \text{otherwise} \end{cases} \]
Using the Lagrange polynomials
\[ L_1^0 = \frac{x - 0}{8 - 0} = \frac{x}{8} \]
\[ L_0^1 = \frac{x - 16}{8 - 16} = \frac{16 - x}{8} \]
the \( \Phi_1(x) \) basis function is defined as
\[ \Phi_1(x) = \begin{cases} \frac{x}{8} & x \in [0, 8] \\ \frac{16 - x}{8} & x \in [8, 16] \\ 0 & \text{otherwise} \end{cases} \]
Using the Lagrange polynomial
\[ L_1^1 = \frac{x - 8}{16 - 8} = \frac{x - 8}{8} \]
the \( \Phi_2(x) \) basis function is defined as
\[ \Phi_2(x) = \begin{cases} \frac{x - 8}{8} & x \in [8, 16] \\ 0 & \text{otherwise} \end{cases} \]
Finally, the linear Spline can be written as:
\[ S(x) = f(x_0)\Phi_0(x) + f(x_1)\Phi_1(x) + f(x_2)\Phi_2(x) = 67\Phi_0(x) + 98\Phi_1(x) + 149\Phi_2(x) \]
c) The error term, in this case, is
\[ E(\beta) = \sum_{i=0}^{n} \left[ f(x_i) - e^{\beta x_i} \right]^2, \]
and we want to solve the following minimization problem
\[ \min_{\beta} E(\beta) \]
Thus, the \( \beta \) parameter is determined imposing
\[ \frac{\partial E}{\partial \beta} = \sum_{i=0}^{n} -2e^{\beta x_i} x_i \left[ f(x_i) - e^{\beta x_i} \right] = 0 \]
After some algebra it reads,
\[ \left( \sum_{i=0}^{n} e^{2\beta x_i} x_i \right) - \left( \sum_{i=0}^{n} x_i e^{\beta x_i} f(x_i) \right) = 0 \]
therefore, substituting the given data and simplifying, the following equation is found
\[ -784 - 2376e^{8\beta} + 16e^{24\beta} = 0. \]

3. [3.5 points] Denote by \( I = \int_{0}^{8} e^{-x^2} \, dx \). Answer the following questions

a) [1 point] Approximate \( I \) using the composite Simpson method with \( n = 2 \).

b) [1 point] Approximate \( I \) using the composite Trapezoidal method with \( n = 4 \).

c) [0.5 points] Is it possible to obtain the exact value of the proposed integral using Simpson or Trapezoidal composite methods? It is necessary to justify the answer.

d) [1 point] When using the Trapezoidal rule, how many intervals, \( n \), are necessary to approximate \( I \) with an absolute error (in absolute value) smaller than 0.001?

*Hint: \( E_{TC}(h) = -f''(\mu)\frac{(b-a)}{12}h^2 \)*

**Solution**
a) As \(a = 0, b = 8\), and \(n = 2\) then \(h = \frac{b-a}{n} = 4\) and \(x_0 = 0, x_1 = 4, x_{1,2} = 6, x_2 = 8\). Applying Simpson’s rule

\[
I \approx \frac{h}{6} (f(x_0) + 2f(x_1) + f(x_2) + 4(f(x_{0,1}) + f(x_{1,2}))) = \frac{4}{6}(f(0) + 2f(4) + f(8) + 4(f(2) + f(6))) = 0.7155
\]

b) As \(a = 0, b = 8\), and \(n = 4\) then \(h = \frac{b-a}{n} = 2\) and \(x_0 = 0, x_1 = 2, x_2 = 4, x_3 = 6, x_4 = 8\). Applying Trapezoidal rule

\[
I \approx \frac{h}{2} (f(x_0) + 2(f(x_1) + f(x_2) + f(x_3)) + f(x_4)) = \frac{2}{2}(f(0) + 2(f(2) + f(4) + f(6)) + f(8)) = 1.0366
\]

c) No, because the function to be integrated is not a polynomial.

d) We are looking for the value \(n\) such that

\[
|E_{TC}(h)| = |\begin{array}{c} |f''(\xi)| (b-a) \\ \frac{1}{12} \end{array} \frac{1}{n^2} | < 0.001
\]

Thus, the value \(|f''(\xi)|\) must be bounded. Let’s start computing the second derivative of the function,

\[
f(x) = e^{-x^2}
\]

\[
f'(x) = -2xe^{-x^2}
\]

\[
f''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = (4x^2 - 2)e^{-x^2}
\]

So,

\[
|f''(x)| = |(4x^2 - 2)||e^{-x^2}|
\]

On one hand, the function \((4x^2 - 2)\) is increasing in the interval \([0, 8]\). Therefore, its maximum is achieved at \(x = 8\). On the other hand, the function \(e^{-x^2}\) is decreasing in the interval \([0, 8]\). Therefore its maximum is achieved at \(x = 0\). Thus, the second derivative can be bounded by

\[
|f''(x)| \leq 254
\]

when \(\xi \in [0, 8]\).

Using the obtained bound for the second derivative and isolating \(n\) from equation ?? we obtain:

\[
\sqrt{\frac{254(8)^3}{12} \left( \frac{1}{0.001} \right)} < n.
\]

Performing the calculations,

\[
3292 < n.
\]

Thus, at least 3293 intervals are necessary to obtain an error smaller than 0.001.